Variations on the (eternal) theme of analytic continuation

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Abstract

This article represents an expanded version of various talks the author has given to a variety of audiences. It contains no new results and its sole purpose is to "wet" the reader's appetite to topics discussed here and provide sufficient guidance and references to the literature.

> Between two truths of the real domain, the easiest and shortest path quite often lies through the complex domain.
>
> P. Painlevé

1 Introduction

Let's illustrate the above quote with a very simple example.

(i) Consider a perfect "bell-shaped graph" $f(x) = \frac{1}{1+x^2}$. If we take its Taylor series around the origin

$$f(x) = \sum_{0}^{\infty} (-1)^n x^{2n},$$

we note immediately that it diverges for all real $x : |x| \geq 1$. Why? Of course, the answer is clear, if we replace x by a complex variable z, the function $f(z) := \frac{1}{1+z^2}$ has two polar singularities at $z = \pm i$ on the boundary of the circle of convergence. Thus, since the Taylor series naturally converge in disks in the complex plane, presence of complex singularities interferes with the behavior of the series in the real domain.

(ii) In the opposite direction if we consider the Taylor series

$$f(x) = \sum_{0}^{\infty} \cos\left(\sqrt{n}\right) x^n$$

that converges only for $|x| < 1$, the function $f(x)$ extends as a smooth, in fact a real-analytic function, to all real $x < 1$. In fact, if we consider $f(z) := \sum\limits_{0}^{\infty} \cos\left(\sqrt{n}\right) z^n$ as an analytic function of the complex variable z, it extends as an analytic function to the whole complex plane $\mathbb{C} \setminus \{1\}$ except one point $z = 1$, where $f(z)$ has an essential singularity. How do we know that?

There is an enormous amount of literature dedicated to studying the properties of analytic functions at large encoded in their local expansions in Taylor series — cf., the classical monographs by P. Dienes and L. Bieberbach ([3], [4, Ch. X]). One of the first results in this direction is the following beautiful theorem of L. Kronecker ([4, Ch. X]).

Theorem 1.1 (L. Kronecker, 1881 [12])**.** *The Taylor series* $\sum\limits_{0}^{\infty} a_n z^n$ *represents a rational function* $f(z) = P(z)/Q(z)$*,* P, Q *are polynomials and* $\max(\deg P, \deg Q) = N$ *if and only if all the determinants*

$$\det \begin{pmatrix} a_0 \cdots a_n \\ a_1 \cdots a_{n+1} \\ \vdots \\ a_n \cdots a_{2n} \end{pmatrix} = 0 \textit{ for all } n \geq N.$$

Our example (i) illustrates this theorem with $N = 2$. Since rational functions are obviously globally defined on the whole Riemann sphere, Kronecker's theorem is a very good example of a mandatory analytic continuation (single-valued as well) of a locally defined Taylor series.

The example (ii) is an illustration of a compilation of results of L. Leau — 1899, S. Wigert — 1900 and G. Faber — 1903, cf. [4, p. 337 ff].

Theorem 1.2 (G. Faber, L. Leau, S. Wigert)**.** *The Taylor series* $\sum\limits_0^\infty a_n z^n$ *with the radius of convergence* 1 *extends to* $\mathbb{C} \setminus \{1\}$ *if and only if there exists a (unique) entire function* $g(z)$ *of order zero* (minimal type) such that $a_n = g(n)$, $n = 1, 2, \ldots$. If $g(z)$ is a polynomial of degree m then 1 is a pole of order $m + 1$.

(Recall that an entire function g is called of minimal type if for every $\varepsilon > 0$ there exists a constant C_ε such that $|f(z)| \le C_\varepsilon e^{\varepsilon |z|}$.)

For example, the geometric series $\sum\limits_0^\infty z^n = \frac{1}{1-z}$ illustrates the latter part of the theorem with $m = 0$.

To illustrate the ideas behind this and similar results let's sketch a slightly more modern result of this type due to T. Qian ([14]) and D. Khavinson [8], (the latter with a different, much shorter proof).

Theorem 1.3 ([8, 14])**.** *Let* $f(z) = \sum\limits_1^\infty b_n z^n$ *be an analytic function in* $\mathbb{D} \coloneqq \{|z| < 1\}$*, and* $b_n = g(n)$*, where* g *is of minimal type in the sector* $S_\varphi \coloneqq \left\{z : |\arg z| < \varphi, 0 < \varphi \le \frac{\pi}{2}\right\}$*. Then,* $f(z)$ *extends to the "heart-shaped" domain* $\Omega_\varphi \coloneqq \left\{z = re^{i\theta}, 2\pi - \cot\varphi \cdot \log r > \theta > \cot\varphi \cdot \log r\right\}$*.*

Note that when $\varphi = \frac{\pi}{2}$, $\Omega_\varphi = \mathbb{C} \setminus [1, +\infty)$.

Sketch of the proof [8] following ideas of Le Roy and Lindelöf [4, p. 340 ff]. By the residue theorem

$$\sum_1^N g(n) z^n = \int\limits_\gamma \frac{g(w) z^w dw}{e^{2\pi i w} - 1}, \tag{1.1}$$

where γ is any contour in S_φ enclosing the integers $1, \ldots, N$ and no others. (Can choose, e.g., for γ the boundary of the sector $\left\{w : |\arg(w - \alpha)| \le \varphi, |w - \alpha| \le R, 0 < \alpha < \frac{1}{2}, R = N + \frac{1}{2}\right\}$.) $\gamma = \gamma_\varphi \cup \gamma_R$, where γ_R is a circular part of the boundary, while γ_φ comprises two sides of the angle with vertex at α.

An elementary argument yields that for all $w \in \mathbb{C} \setminus \bigcup\limits_{n=-\infty}^\infty \{z : |z - n| < \eta, \eta > 0\text{-small}\}$, $|e^{2\pi i w} - 1| \ge c > 0$, $c = c(\eta)$ is a constant ([4, p. 341]).

Hence, the part of the integral (1.1) restricted to γ_R tends to zero when $R \to \infty$ (i.e., $N \to \infty$) for $z = -r$, $0 < r < 1$, i.e., z in $\mathbb{D}$ and on the negative radius. (This is seen from an elementary estimate $|e^{\pi iw} - e^{-\pi iw}| \geq ce^{\pi R|\sin\theta|} > c$, for $w = \alpha + Re^{i\theta}$, assuming at first $\varphi < \frac{\pi}{\alpha}$.) Thus, for $z \in (-1, 0]$ in $\mathbb{D}$,

$$f(z) = \int\limits_{\Gamma_\varphi} \frac{g(w)z^w}{e^{2\pi i w} - 1}\, dw, \tag{1.2}$$

where $\Gamma_\varphi \coloneqq \lim\limits_{R\to\infty} \gamma_\varphi = \{w \cdot \arg(w - \alpha) = \pm\varphi\}$. Thus, in order for the integrand in (1.2) to decay exponentially on Γ_φ, we obtain using the assumptions on g, that it suffices to have

$$\cot\varphi \cdot \log|z| < \arg z, \text{ for } \arg(w - \alpha) = \varphi$$

and

$$2\pi - \cot\varphi \cdot \log|z| > \arg z \text{ for } \arg(w - \alpha) = -\varphi.$$

This proves the statement for $\varphi < \frac{\pi}{2}$, since (1.2), an analytic function in z, converges for all $z \in \Omega_\varphi$ and coincides with f on $(-1, 0)$. Finally, $\Omega_{\frac{\pi}{2}} = \bigcup\limits_{\varphi<\frac{\pi}{2}} \Omega_\varphi$. $\square$

Remark. The assumptions on g can be relaxed somewhat further (to g being of exponential type less than π — cf. [4, pp. 341–342]). It is not known whether Ω_φ is the largest domain one can extend $f(z)$ to. Thus, the "only if" part is still missing unlike for classical results of Kronecker and Leau (Thms. 1.1 and 1.2).

Of course, it is impossible in an article like this one to survey all beautiful topics investigated in the classical avenue of analytic continuation: monodromy, continuation of algebraic functions, over-convergence and gap series, universal Taylor series, and many others. We hope that an interested reader will be tempted to continue research on her own: [4, Ch. X, XI], [3] are good books to start. The more recent vast literature on universal Taylor series can be found on MathSciNet.

From these classical themes of continuation of Taylor series, let's make a leap to the problem of analytic continuation of solutions of most basic equations of mathematical physics.

2 Continuation of solutions of linear PDE

2.1 ODE vs. PDE

It is well known that the solution of the initial value problem for the linear ODE

$$\begin{gathered} w^{(n)}(z) + a_{n-1}(z)w^{(n-1)}(z) + \cdots + a_n(z)w'(z) + a_0(z)w(z) = f(z), \\ w(0) = w_0, \ldots, w^{(n-1)}(0) = w_{n-1}, \end{gathered} \tag{2.1}$$

with the coefficients a_j's and f analytic in a domain Ω containing the origin, extends as analytic function throughout Ω. (It might end up being multi-valued, if Ω is not simply connected, i.e., has holes, but nevertheless can be analytically continued everywhere in Ω.)

Yet, if we consider a very simple initial value problem for PDE:

$$\frac{\partial w}{\partial y} = x^2 \frac{\partial w}{\partial x}, \quad w(x,0) = x; \tag{2.2}$$

the easily found solution $w := \frac{x}{1-xy}$ blows up arbitrarily close to the initial line $\{y = 0\}$ on the hyperbola $y = 1/x$. How can we explain this?

Moreover, if we consider a more general initial value problem for (2.2) with arbitrary "data" $w(x,0) = f(x)$, where f is a polynomial, or an entire function, one easily checks that the solution is

$$w(x,y) := f\left(\frac{x}{1-xy}\right).$$

This is striking, since it yields that the variety $\Gamma := \{xy = 1\}$ is the only possible carrier of singularities for all solutions to (2.2), independently of the data, as long as the data itself has no singularities.

Let's postpone the heuristic explanation of this fact till later and discuss another natural problem of analytic continuation coming from mathematical physics.

2.2 G. Herglotz' Memoir of 1914

The following question was first tackled by G. Herglotz in 1914 (also, cf. [15, 18]). Imagine a solid in $\mathbb{R}^3$, or a "plate" (a domain) in $\mathbb{R}^2$, Ω, bounded by, say, a nice algebraic surface (or, a curve).

Let

$$u_\Omega(x) = \int\limits_\Omega k_n(x,y)\,dy, \tag{2.3}$$

where

$$\begin{aligned} k_n(x,y) &= \frac{1}{2\pi}\log\frac{1}{|x-y|},\ n=2 \\ &= \frac{-1}{4\pi}\,\frac{1}{|x-y|},\ n=3 \end{aligned}$$

be the "gravitational" or "electrostatic" potential of Ω. Obviously, u is harmonic outside Ω and the natural question tackled by Herglotz was: how far can one harmonically continue $u(x)$ inside Ω before running into a singular point? (Herglotz did answer the question in $\mathbb{R}^2$ and made some headway in $\mathbb{R}^3$, but because his prize winning memoir appeared on the brink of World War I it fell into oblivion while most of the results were rediscovered by other authors — cf. the references in [10, 16]. For example, if Ω is a disk or a ball, then, of course, (e.g., $n = 3$) by the mean value theorem, we have:

$$u_\Omega(x) = -\frac{1}{4\pi}\int\limits_{\{|y|<1\}}\frac{dy}{|x-y|} = \frac{\text{const}}{|x|}$$

(dy, of course, stands for the Lebesgue measure in $\mathbb{R}^n$.)

So, $u_\Omega(x)$ extends as a harmonic function everywhere in $\mathbb{R}^3$ except for the center of the ball. This goes back to I. Newton and is well-known. What is perhaps less well-known is that conclusion stays true for

$$u_{\Omega,p}(x) = -\frac{1}{4\pi}\int\limits_\Omega\frac{p(y)\,dy}{|x-y|}, \quad \Omega = \{y : |y| < 1\}$$

with an ARBITRARY polynomial, or even an entire density $p(y)$. In the latter case, all the symmetry associated with the ball goes out the window, and the conclusion that $u_{\Omega,p}$ extends harmonically to all of $\mathbb{R}^3\setminus\{0\}$, matches Leau's theorem from Section 1 in mystery and beauty.

Herglotz' problem is often restated in more "physical" terms: consider the exterior gravitational potential of an analytic mass density $p_0(y)$ in the region Ω. Find a smaller object E inside Ω and a different mass-density p_1 on E that is gravi-equivalent to p_0, i.e., such that the potential u_{E,p_1} and u_{Ω,p_0} coincide outside of Ω.

Example 2.1. For $\Omega = \{x : |x| < 1\} \subset \mathbb{R}^3$ (or, more generally, $\mathbb{R}^n$) $p_0 =$ polynomial of degree $\leq N$, $E = \{0\}$ and p_1 is the distribution of order $\leq N$ at the origin.

Example 2.2 (cf., [9, 10, 16])**.** An oblate spheroid

$$\Omega := \left\{ x \in \mathbb{R}^3 : \frac{x_1^2}{a} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} \leq 1, a > b > 0 \right\}$$

(planet Earth, e.g.). Then, for say, uniform density $p_0 \equiv 1$ (or any other polynomial, or, entire density) $u_\Omega(x)$ extends into $\Omega \setminus E$, where

$$E := \left\{ x_3 = 0, x_1^2 + x_2^2 \leq a^2 - b^2 \right\}$$

is the caustic disc. The relevant density p_1 on E (relevant to $p_0 = 1$) is algebraic and equals const $\left(a^2 - b^2 - x_1^2 - x_2^2\right)^{1/2}$ — cf. [10, Ch. 15].

Example 2.3. A prolate spheroid $\Omega := \left\{ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{b^2} \leq 1, a > b > 0 \right\}$ gives a completely different picture, an exciting mystery on its own. The potential u_Ω extends to $\Omega \setminus E$, but E in this case is a 1-dimensional segment $\left\{ x_2 = x_3 = 0, |x_1| \leq \sqrt{a^2 - b^2} \right\}$, while the density $p_1 = (a^2 - b^2 - x_1^2)$ is a polynomial. Moreover, below we shall touch upon the rather deep problem regarding the dramatic differences in singularities of u_Ω in the latter two examples: bounded, a square-root type singularity in the former, and unbounded — in the latter.

2.3 A further discussion of the Herglotz question

As one readily obtains directly from (2.3) via Green's theorem,

$$\Delta u_\Omega = \chi_\Omega, \tag{2.4}$$

where $\chi_\Omega(x) = \begin{cases} 1, & x \in \Omega \\ 0, & x \in \mathbb{R} \setminus \overline{\Omega} \end{cases}$ and stands for the characteristic function of Ω. Denote by M, the so-called modified Schwarz potential (of $\partial\Omega$) the solution of the following initial value problem

$$\begin{gathered} \Delta M = 1 \text{ near } \Gamma := \partial\Omega; \\ M = \nabla M = 0 \text{ on } \Gamma. \end{gathered} \tag{2.5}$$

(The solution exists and is unique by the Cauchy–Kovalevskaya theorem — cf. [10], e.g.) Then, the function

$$u := \begin{cases} u_\Omega, & \text{outside } \Omega \\ u_\Omega - M & \text{inside } \Omega \end{cases} \tag{2.6}$$

gives the desired continuation. Indeed, $u_\Omega - M$ is harmonic in Ω near Γ and coincides with u_Ω on Γ together with its first derivatives. The statement then follows by a straightforward application of Green's formula — cf. [10, Thm. 6.1].

For an arbitrary polynomial, or entire mass density p we only need to modify (2.5) and define M_p as a solution of the initial value problem

$$\begin{cases} \Delta M_p = p \text{ near } \Gamma; \\ M_p = \nabla M_p = 0 \text{ on } \Gamma. \end{cases} \tag{2.7}$$

If instead of M we consider the Schwarz potential u_Γ of Γ defined by

$$\begin{cases} \Delta u_\Gamma = 0 & \text{near } \Gamma; \\ u_\Gamma = \frac{1}{2n}\,|x|^2 \text{ on } \Gamma; & \operatorname{grad} u_\Gamma = \frac{1}{n}\,\vec{x} \text{ on } \Gamma \end{cases} \tag{2.8}$$

($n = 2$ or 3, as in our examples), then obviously, $M_\Gamma = \frac{1}{2n}\,|x|^2 - u_\Gamma$. Similarly, $M_p = Q - u_{\Gamma,p}$, where Q is a polynomial, an entire function such that $\Delta Q = p$, and u_Γ accordingly defined as a solution of the initial value problem similar to (2.8):

$$\begin{cases} \Delta u_{\Gamma,p} = 0 \text{ near } \Gamma; \\ u_{\Gamma,p} = Q, \nabla u_{\Gamma,p} = \nabla Q \text{ on } \Gamma. \end{cases} \tag{2.9}$$

Thus, if we could show that the singularities of any initial value problem for the Laplace operator posed on Γ are only dictated by Γ itself, not by initial data ($\frac{1}{2n}\,|x|^2$, or Q), we would have achieved the high ground needed for understanding the Herglotz' problem. A deep and beautiful theory of Leray explains the origins for appearance of singularities of initial value problems near initial surfaces — cf. [10, Ch. 13, 19–20] and Leray's original papers referenced there.

Indeed, generically, it asserts that the singularities appear and take off (locally, sic!), from the initial surfaces at the same places and along the same routs independently of data. Moreover, in dimension 2, and also, in higher dimensions, but only for quadratic surfaces, it has been proved that the local theory of Leray, first verified only near initial surfaces, holds globally — cf. [10]. Here, we will simply illustrate the Leray principle by a couple of straightforward examples. We only sketch the main steps, more details can be found in [10].

Example 2.4. Let $\Omega = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 < 0, a > b \right\}$ be an ellipse, $\Gamma := \partial\Omega$.

One can calculate u_Γ, and then further

$$\frac{1}{2}\nabla u_\Gamma = \frac{\partial u_\Gamma}{\partial \bar{z}} = \frac{a^2+b^2}{a^2-b^2}\,\bar{z} - \frac{2ab}{a^2-b^2}\sqrt{\bar{z}^2-c^2}, \quad c^2 - a^2 - b^2 \tag{2.10}$$

($z = x + iy$, as usual). So, the singularities of u_Γ are at the foci of Ω. The solution of the initial value problem (2.8) is fine by the Cauchy–Kovalevskaya theorem near complexified quadratic curve $\widehat{\Gamma}$ in $\mathbb{C}^2$,

$$\widehat{\Gamma} := \left\{ x, y \in \mathbb{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \right\}$$

except for 4 points $\left\{ \left(\pm\frac{a^2}{c}, \pm\frac{b^2}{c} \right) \right\}$ on $\widehat{\Gamma}$ where C–K theorem breaks down. From those "bad" points, as Leray's theory asserts, the singularities travel along 4 complex (characteristic) lines $\{(x,y) \in \mathbb{C}^2 : x \pm iy = \text{const}\}$ tangent to $\widehat{\Gamma}$ at the above characteristic points. These lines, the"carriers" of singularities, reach the "real" space $\mathbb{R}^2$ at the foci $(\pm c, 0)$ of the ellipse. Since the carries of singularities depend on Γ only, logarithmic potentials of Ω with arbitrary polynomial, or entire densities exhibit the same behaviour and might become singular only at the foci as well.

Moreover, when $b \uparrow a$, $c \downarrow 0$, an ellipse becomes a circle, the "bad" points on $\widehat{\Gamma}$ all move to infinity and the singularities of (2.10) change from algebraic ($\sqrt{\ }$-type) to polar-$\frac{c}{z}$ at the origin, the limiting position of the collapsing foci.

A similar, but technically much more demanding analysis provides the justification for Examples 2.1–2.3 in $\mathbb{R}^3$, and in general in $\mathbb{R}^n$ — cf. [10] and the works of G. Johnsson referenced therein. However, we emphasize that for algebraic surfaces of degree ≥ 3 in $\mathbb{R}^n$, $n \geq 3$, the analysis of Herglotz' problem, i.e., the global version of Leray's principle, is still waiting to be discovered.

We shall finish this section with another transparent example illustrating Leray's theory.

Example 2.5. Consider the initial value problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial x \partial y} = 0, \text{ near } \Gamma := \left\{y = x^3\right\};\\ &\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x \text{ on } \Gamma. \end{aligned} \tag{2.11}$$

One readily finds the solution $u(x,y) = \frac{x^4}{4} + \frac{3y^{4/3}}{4}$ that is "ramified" around $\{y = 0\}$. The latter is in fact Leray's characteristic tangent to Γ at the (unique, w.r.t. $\frac{\partial^2}{\partial x \partial y}$ operator) "bad", characteristic point $(0,0)$.

In higher dimensions the situation is more complicated. In a nutshell, there are more "complex characteristic lines" tangent to the initial surface that we view as continuation of Γ into $\mathbb{C}^n$. These lines carry out singularities off the initial surface. The analytic functions having singularities on a piece of an analytic hypersurface however must be singular on the whole hypersurface by a celebrated theorem of Hartogs (in $\mathbb{C}^n$, $n > 1$, of course). In other words, the singularities propagate from "bad" points on the complexified (embedded into $\mathbb{C}^n$) surface Γ and then exhibit themselves in $\mathbb{R}^n$ at points where the Leray characteristic tangent, the carrier of singularities, hit $\mathbb{R}^n$. This is transparent and proved rigorously (cf. [10, Ch. 13], e.g.), by G. Johnsson for quadratic surfaces in $\mathbb{R}^n$ — cf. [10, Ch. 19-20] and references to Johnsson's original papers contained therein. This also explains the difference in the nature of singularities in Examples (2.2)–(2.3). In the case of the oblate spheroid, each point on the circular caustic $\{x_1^2 + x_2^2 \leq a^2 - b^2, x_3 = 0\}$ is a "meeting point" of true characteristic lines coming from $\mathbb{C}^3$ and tangent at characteristic points on the complexified surface $\left\{(x_1, x_2, x_3) \in \mathbb{C}^3 : \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1 = 0, a > b\right\}$. For the prolate spheroid, each point of the caustic segment $\left\{|x_1| \leq \sqrt{a^2 - b^2}, x_2 = x_3 = 0\right\}$ is a meeting point of infinitely many characteristics, thus causing unbounded singularities. So, intuitively, the idea that "more carriers of singularities meeting at a point in $\mathbb{R}^n$" should result in a "heavier" singular behaviour is tempting and reasonable. However, essentially, nothing has been rigorously proved along these lines. A worthy and challenging avenue for further research.

3 Analytic Continuation and Problems of Uniqueness

Consider the spherical shell $\Omega := \{x \in \mathbb{R}^3 : r < |x| < R\}$ ($|x|^2 = x_1^2 + x_2^2 + x_3^2$, as usual). Let u be a harmonic function in Ω that vanishes on the segment $(-R, -r)$ of the x_1-axis. Then, let us pose the question (cf. [10, Ch. 9]).

Question 3.1. Must u also vanish on the segment $r < x_1 < R$, $x_2 = x_3 = 0$?

The same question in 2 dimensions can be easily settled by elementary complex analysis. To fix the ideas, let $r = 1$, $R = 2$, $n = 2$. Consider the harmonic function $v(z) := u(z) + u(\bar{z})$ in the annulus $\Omega := \{1 < |z| < 2\}$. By the Schwarz effection principle, $v \equiv 0$ in a small disk centered on $(-2, -1)$, say, $\left\{z : \left|z + \frac{3}{2}\right| < \frac{1}{4}\right\}$. Hence, $v \equiv 0$ in Ω and, accordingly, $2u(x) = v(x) = 0$ on $(1, 2)$. Thus, in this situation the answer is "yes". Although, the answer is also "yes" in $\mathbb{R}^n$, $n \geq 3$, the above argument of course, fails. Moreover, the above argument doesn't work either if instead of a line through the center of the annulus, or a spherical shell, we consider an arbitrary line still cutting Ω in two disjoint segments. It might come as a surprise that the answer remains "yes" if $\frac{R}{r} > 3$ (a thick annulus, or shell), but becomes "no, not necessarily" if $\frac{R}{r} \leq 3$ (a thin annulus) and the constant 3 is sharp (cf. [10, Ch. 9], [11]). Even more intriguing ([10, Ch. 9]), the same question posed for a torus in $\mathbb{R}^3$, has always a negative answer in general.

What is the high ground for this question? The answer is: analytic continuation. Indeed, a harmonic in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, function u automatically extends as a holomorphic function of n variables to a domain $\widehat{\Omega}$ in $\mathbb{C}^n$. $\widehat{\Omega}$ can be viewed in a rather simple way as follows. For all $x^o \in \mathbb{R}^n \setminus \Omega$, consider the isotropic cone $\Gamma_{x^o} := \left\{z \in \mathbb{C}^n : \sum_1^n \left(z_j - x_j^0\right)^2 = 0\right\}$. Then, $\widehat{\Omega} := \mathbb{C}^n \setminus \bigcup_{x^o \in \mathbb{R}^n \setminus \Omega} \Gamma_{x^o}$ — cf. [1, Ch. 1].

The beautiful fact established in the theory of linear analytic PDE is that solutions of ALL PDEs ($\Delta^n +$ lower terms) $u = f$ with, say polynomial or entire coefficients in Ω automatically extend to $\widehat{\Omega}$. For example, if $\Omega = \{|x| < 1\}$ is the unit ball in $\mathbb{R}^n$, $n \geq 2$, $\widehat{\Omega} = \left\{z \in \mathbb{C}^n : \left(\|z\|^4 - \left|\sum_1^n z_j^2\right|^2\right)^{1/2} + \|z\|^2 < 1\right\}$, the celebrated Lie ball (cf. [1, 2, 10]). (For $n = 2$, $\widehat{\Omega} = \{(X, Y) \in \mathbb{C}^2 : |X \pm iY|\}$, the bidisk (cf. [1, 2, 10]). Thus, a sufficient condition that would yield an affirmative answer to our question, is whether the intersection $\widehat{\Omega} \cap \{Y = c\}$ the harmonicity hull of the shell Ω and the complexified line $\{y = c\}$ (in dimension 2, e.g.) is connected or not. In the original question for the annulus, for example, this intersection becomes $\{(X, c) : r < |X \pm ic| < R\}$ and is disconnected if $\frac{R-r}{2} < c < r$, and connected if $0 < c \leq \frac{R-r}{2}$. If $\frac{R}{r} \geq 3$, e.g., $\frac{R-r}{2} \geq r > c$, so the intersection of $\widehat{\Omega}$ and the complex line $\{Y = c\}$ is connected. The fact that the constant 3 is sharp is seen (T. Ransford) by taking Ω to be an annulus separating $\{0, -i\}$ from i, and $u(z) := \operatorname{Re} \sqrt{z(z-i)(z+i)}$, where we can take any branch of the square root. Then, $u(x) = 0$, $x < 0$ and $u(x) > 0$, $x > 0$. $\frac{R}{r} < 3$ but can be made arbitrary close to 3 — see [11], [10, Ch. 9] for more details.

As is remarked in [11], this simple consideration allows to answer questions similar to Q. 3.1 not only for solutions of linear PDE with the power of the Laplacian in the principle part, but also for functions represented by arbitrary Riesz potentials, the latter, in general, need not satisfy any linear PDE.

4 Analytic Continuation of Series of Zonal Harmonics and Series of Orthogonal Polynomials

By analogy with analytic continuation of Taylor series, let's consider the problem of finding singularities of other series expansions. To fix the ideas, let

$$u := \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta) \tag{4.1}$$

be an axially symmetric harmonic function in the unit ball in $\mathbb{R}^3$. $a_n \in \mathbb{R}$, $\lim_{n\to\infty} |a_n|^{1/n} = 1$, $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2-1)^n\right]$ are Legendre polynomials (orthogonal on $[-1, 1]$, $\|P_n\|_2^2 = \frac{2}{2n+1}$). $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ the distance to the origin, θ is the usual azimuth angle in spherical coordinates. One can easily verify that the expansion (4.1) diverges for $r > 1$, so u must have singularities on the unit sphere $S^2 := \{r = 1\}$. The question is where? The following remarkable theorem was proved by G. Szegő in 1954 [17].

Theorem 4.1. *$u(r,\theta)$ extends harmonically across the circle $(1,\theta_0 : 0 \le \varphi \le 2\pi)$ on the sphere S^2 if and only if the Taylor series* (!) $f(\xi) \coloneqq \sum_{k=0}^{\infty} a_n \xi^n$ extends across $\xi_0 \coloneqq e^{i\theta_o}$.

This is a truly amazing result, since at first glance, the expansions in zonal harmonics and the Taylor series built with the same coefficients should have nothing in common. Moreover, inspired by Szegő's theorem, Nehari proved the following beautiful follow-up [13].

Theorem 4.2. *Let $\{a_n\} \in \mathbb{C}$, satisfy $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \frac{1}{R}$, $R > 1$ and let $f(t) = \sum_{n=0}^{\infty} a_n P_n(t)$ be (as is easily checked) an analytic function inside the ellipse $\mathcal{D}_R$ with foci at ± 1 and sum of whose semiaxes equals R. In other words, $\mathcal{D}_R \coloneqq \left\{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, a+b = R, a-b = \frac{1}{R}\right\}$. $P(t)$, as in Szegő's theorem, denote Legendre polynomials. Then, $f(t)$ is analytically continuable across $t_o \in \partial\mathcal{D}_R$ if and only if the analytic function $g(s) \coloneqq \sum_{n=0}^{\infty} a_n s^n$, $|s| < R$, where s and t are related by the conformal map $s = \varphi(t) = t + \sqrt{t^2-1}$, $\varphi : \widehat{\mathbb{C}} \setminus [-1,1] \to \{s \in \mathbb{C} : |s| > 1\}$, $\varphi(\infty) = \infty$, is analytically continuable across the corresponding point $s_0 = \varphi(t_0)$, $\left(t_0 = \frac{1}{2}\left(s_0 + s_0^{-1}\right)\right)$, $s_0 \in \{s : |s| = R\}$.*

Once again, the reader should observe how Nehari's result, unexpectedly, connects the singularities of the expansion in orthogonal polynomials with seemingly disjoint Taylor series.

Since there is a wide variety of results allowing one to identify singularities of Taylor series on the circle of convergence, the above two results provide a powerful tool for identifying singularities of harmonic functions and orthogonal polynomial expansions.

Both theorems can be significantly extended by replacing Legendre polynomials with arbitrary Jacobi orthogonal polynomials (R. P. Gilbert (1969), P. Ebenfelt–D. Khavinson–H. S. Shapiro (1996)) — cf. [10, Ch. 10], [5], [6] and references therein.

Recall that for $\alpha, \beta > -1$, the Jacobi polynomials are orthogonal polynomials on $[-1,1]$ with respect to the weight $(1-x)^\alpha(1+x)^\beta$, normalized by $P_n^{\alpha,\beta}(1) = \binom{n+\alpha}{n}$. So, $\alpha = \beta = 0$ corresponds to Legendre polynomials,

$\alpha = \beta = -\frac{1}{2}$ — Tschebysheff polynomials (that, in turn, under a suitable change of variables, correspond to monomials z^n, $n \geq 0$ in the unit disk — cf. [5, 6, 10]; $\alpha = \beta = \frac{k-3}{2}$, $k \geq 2$ being an integer corresponds to ultraspherical polynomials appearing in expansions of axially symmetric harmonic functions in $\mathbb{R}^k$ (Gegenbauer polynomials).

Now, the original proofs of Thms. 4.1 and 4.2 boil down to writing down the given expansions in terms of a certain integral and ingenious manipulation of the latter. The "high ground" approach advocated and developed in [5, 6] consists of noticing that ALL relevant expansions in general Jacobi polynomials can be interpreted as solutions of a Cauchy problem for a linear PDE in two variables with the same initial data. Of course, the partial differential operator corresponding to every particular expansion is different in each case. But all of them share the same principal part, the part of the differential operator that involves the senior derivatives. A deep result in the theory of linear PDE, based on the 1970 extension by M. Zerner (cf. [10, Ch. 4] of the classical Cauchy–Kovalevskaya theorem yields that the singularities of the solutions of the Cauchy problems locally depend exclusively on the principal part of the differential operator. Hence, all the expansions in Thms. 4.1 and 4.2 share the same singularities thus unveiling the mystery behind Szegő's and Nehari's results.

5 An Epilogue

This article's only intent is to initiate for the curious reader a few possible modern directions in the classical theme of analytic continuation. There is absolutely no way to cover all possible topics, thus our choices were limited to several topics the author felt most comfortable with.

There are so many themes which were left out entirely, e.g., beautiful results of Eisenstein regarding algebraic properties of Taylor series depending on properties of coefficients — [3, 4]. Essentially, no deep and subtle results, starting from Painlevé classical researchers, dealing with classification of singularities and analytic continuation of solutions to nonlinear ODE in the complex domain — cf., e.g., [7]. In classical potential theory, we left out beautiful modern generalizations due to A. Givental of the Newton's "no gravity in the ellipsoidal cavity" theorem, Ivory's theorem, MacLauren's mean value theorem for ellipsoids, viewed from the modern viewpoint of analytic continuation of Cauchy's problem for the Laplace equation — cf. [9, 10]

and references therein. The theme of analytic continuation of solutions to the Dirichlet problem in domains with algebraic boundaries is far from developed and has an attractive array of important open problems, even in two dimensions — cf. [10, Ch. 18]. Even more basic open problems await an interested reader if one extends the search for singularities of solutions to the classical Dirichlet problem for the Laplacian to that for Helmholtz' equation. In other words, expanding the program to study possible singularities of the eigenfunctions of the Laplacian in domains with algebraic boundaries. The reigning open conjecture that ellipsoids are the ONLY domains for which *all* eigenfunctions are entire (and of exponential type) remains virtually untouched.

A fairly recent solution by P. Ebenfelt and D. Khavinson — cf. [10, Chs. 11, 12] of the problem of reflection of harmonic functions across analytic hypersurfaces in higher dimensions (or, why doesn't Schwarz reflection principle work in, say, $\mathbb{R}^3$?) opens up a new venue for investigations: the "antenna problem". In short, it is the question of possibility of reflection from a point to a compact set vs. point-to-point reflection.

Once again, more important for applications is the reflection question for solutions of the Helmholtz equation, i.e., the eigenfunctions for the Laplacian. That playing field is widely open as well — cf. [10] and references therein.

Finally, we have only mentioned in passing the powerful methods of analytic continuations of solutions of linear analytic PDE combined with the modern techniques of several complex variables. The results culminate in Leray's theory of propagation of irregularities through $\mathbb{C}^n$. The underlying techniques based on the so-called method of "globalizing families" is both clear and quite powerful — cf. [10, Chs. 4–10, 19, 20] and references therein.

However, at present, the theory is more or less complete (we mean the **global** theory of propagation of singularities) mostly in two variables and also in $n \geq 3$-variables, but there exclusively for singularities initiated on quadratic surfaces [10, Ch. 19–20]. Once again, the importance of the remaining open problems is difficult to overestimate.

In conclusion, by this short survey we wanted to demonstrate that the classical theme of analytic continuation of functions of one variable that has intrigued researchers for at least 200 years since the concept of an analytic function had come into focus, is alive, doing well and is quite rich with plenty of attractive and beautiful problems, conjectures, and attractive routes for further study. Thus, we hope that this small survey and the appended references will prompt the reader to invest time and effort in further research on these truly "eternal" topics.

Acknowledgement

The author gratefully acknowledges support from the Simons Foundation.

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